Global Sensitivity Analysis
Motivation

- Consider a numerical model of a structure

Inputs

- $A$
- $E$
- $f_y$
- $P_2$

Model

Output

- midsnap displacement
- Failure index (0,1)

- Will all the input parameters contribute to the response?
- Which input factors are more influential than others?
Applications of GSA

• To gain insights
  • How different parameters and their interactions affect a system

The parameters related to the local site diminution effect (...) can be in general neglected; (...) temporal envelope function parameters have a considerable contribution towards the total risk only for lower moment magnitudes, especially (...) (Vetter & Taflanidis, 2011)
Applications of GSA

- **To gain insights**
  - How different parameters and their interactions affect a system

- **Dimensionality reduction**
  - By identifying uninfluential (redundant) factors

- **Informed decision making**
  - To find parameters for which new data acquisition reduces target uncertainty the most
  - To identify most effective decision options

- **Model diagnostics**
  - After developing a model, one may compare GSA results with expert knowledge
Local Sensitivity Analysis

- Rate of change (slope)
  \[ S_i^D (X) = \frac{\partial g(X)}{\partial X_i} \]

- Studies the impact of small perturbations on the model outputs
- Evaluated at a reference point \( X \)
- One-factor-at-a-time evaluation
- Used in reliability analysis / optimization

e.g. structural reliability analysis
Local Sensitivity Analysis

- How to explore entire variability space?

- Consider an example

\[ Y = g(X_1, X_2) = 2X_1 + X_2 \]

\[ X_1 \sim N(0, 0.5^2), X_2 \sim N(0, 5^2) \]

Among \(X_1\) and \(X_2\), which variable is more “important”? 

Is the average of gradients a good measure?
Local Sensitivity Analysis

\[ Y = g(X_1, X_2) = 2X_1 + X_2 \]

\[ X_1 \sim N(0,0.5^2), \ X_2 \sim N(0,5^2) \]

• If we decide the importance by ‘partial derivative’ measure, 
  \( X_1 \) is important

• But if we inspect the scatter plots,

\( X_2 \) seem to dominate the response

\[ Y \text{ vs. } X_1 \]

\[ Y \text{ vs. } X_2 \]
Local Sensitivity Analysis

• Sigma-normalized derivative

\[ S_i^{SD}(X) = \frac{\sigma_{X_i}}{\sigma_Y} \frac{\partial g(X)}{\partial X_i} \]

\(X_2\) is five times more important than \(X_1\)
Local Sensitivity Analysis

- ‘Partial derivative’ in the standard random variable domain
  - When the random variables are **independent**, each variable can be transformed to the standard normal variable, $Z_i = T(X_i)$.
  - Example: FORM analysis

Importance vector:
normalized gradient

$$\alpha = -\frac{\nabla G(Z^*)}{\|\nabla G(Z^*)\|}$$

Note

$$\alpha = -\frac{Z^*}{\beta}$$

For dependent variables:

$$Z_1 = F_{Z_1}^{-1}(F_{X_1}(X_1))$$
$$Z_2 = F_{Z_2}^{-1}(F_{X_2|X_1}(X_2|X_1))$$
$$\vdots$$
$$Z_n = F_{Z_n}^{-1}(F_{X_n|X_{n-1},...,X_1}(X_n|X_{n-1},...,X_1))$$
Variance-based Sensitivity

- Intuition behind the Sobol indices

\[ \mathbb{E}_{x_i} [Y| x_i] \text{ is almost constant throughout different } x_i \text{ values} \]
\[ \text{Var}_{x_i} \left[ \mathbb{E}_{x_i} [Y| x_i] \right] \text{ is almost zero} \]
\[ \text{Low sensitivity} \]

\[ \mathbb{E}_{x_j} [Y| x_j] \text{ depends on } x_j \]
\[ \text{Var}_{x_j} \left[ \mathbb{E}_{x_j} [Y| x_j] \right] \text{ is larger} \]
\[ \text{High sensitivity} \]
Variance Decomposition

• $\text{Var}_{x_i} \left[ \mathbb{E}_{x_i} [Y|x_i] \right]$ is a measure of sensitivity

• The Law of Total Variance

$$\text{Var}[Y] = \text{Var}_{x_i} \left[ \mathbb{E}_{x_i} [Y|x_i] \right] + \mathbb{E}_{x_i} \left[ \text{Var}_{x_i} [Y|x_i] \right]$$

- Explained by $x_i$
- Not explained by $x_i$

i.e. the expected reduction in variance that would be obtained if $x_i$ could be fixed
Variance Decomposition

- $\text{Var}_{x_i} \left[ \mathbb{E}_{x_i}[Y|x_i] \right]$ is a measure of sensitivity
- The Law of Total Variance

\[
\text{Var}[Y] = \text{Var}_{x_i} \left[ \mathbb{E}_{x_i}[Y|x_i] \right] + \mathbb{E}_{x_i} \left[ \text{Var}_{x_i}[Y|x_i] \right]
\]

- Derivation

\[
\text{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
= \mathbb{E}_{x_i} \left[ \mathbb{E}_{x_i}[Y^2|x_i] \right] - \mathbb{E}_{x_i} \left[ \mathbb{E}_{x_i}[Y|x_i] \right]^2 \\
= \mathbb{E}_{x_i} \left[ \text{Var}_{x_i}[Y|x_i] + \mathbb{E}_{x_i}[Y|x_i]^2 \right] - \mathbb{E}_{x_i} \left[ \mathbb{E}_{x_i}[Y|x_i] \right]^2 \\
= \mathbb{E}_{x_i} \left[ \text{Var}_{x_i}[Y|x_i] \right] + \mathbb{E}_{x_i} \left[ \mathbb{E}_{x_i}[Y|x_i]^2 \right] - \mathbb{E}_{x_i} \left[ \mathbb{E}_{x_i}[Y|x_i] \right]^2 \\
= \mathbb{E}_{x_i} \left[ \text{Var}_{x_i}[Y|x_i] \right] + \text{Var}_{x_i} \left[ \mathbb{E}_{x_i}[Y|x_i] \right]
\]
Variance Decomposition

- $\text{Var}_{x_i} \left( \mathbb{E}_{x_{\bar{i}}} [Y|x_i] \right)$ is a measure of sensitivity
- The Law of Total Variance

$$\text{Var}[Y] = \text{Var}_{x_i} \left( \mathbb{E}_{x_{\bar{i}}} [Y|x_i] \right) + \mathbb{E}_{x_i} \left[ \text{Var}_{x_{\bar{i}}} [Y|x_i] \right]$$

$$1 = \frac{\text{Var}_{x_i} \left[ \mathbb{E}_{x_{\bar{i}}} [Y|x_i] \right]}{\text{Var}[Y]} + \frac{\mathbb{E}_{x_i} \left[ \text{Var}_{x_{\bar{i}}} [Y|x_i] \right]}{\text{Var}[Y]}$$

Sensitivity index

In range of $[0,1]$
Variance Decomposition

- The Law of Total Variance

\[ 1 = \frac{\text{Var}_{x_i} [\mathbb{E}_{x_i}[Y|x_i]]}{\text{Var}[Y]} + \frac{\mathbb{E}_{x_i} [\text{Var}_{x_i}[Y|x_i]]}{\text{Var}[Y]} \]

- Sobol Sensitivity Index

\[ S_i = \frac{\text{Var}[\mathbb{E}[Y|x_i]]}{\text{Var}[Y]} \]

\[ S_i = 1 - \frac{\mathbb{E}[\text{Var}[Y|x_i]]}{\text{Var}[Y]} \]

- Main-effect index, First-order index
Second-order Sensitivity Measures

\[ S_{ij} = \frac{\text{Var}_{x_i x_j} \left[ \mathbb{E}_{x_{\overline{i} j}} [Y|X_i, X_j] \right]}{\text{Var}[Y]} - S_i - S_j \]

\( S_{ij} \) captures the pure interaction effect

- Contribution of \( X_j \)
- Joint contribution of \( X_i \) and \( X_j \)
- Contribution of \( X_i \)
Interaction Effect

- Interaction effect: $X_i$ vs. $Y$ is affected by $X_j$

\[
g_A(X_1, X_2) = 3X_1^3 + \log(X_2)
\]

\[
g_B(X_1, X_2) = 3X_1^3 + \log(X_2) + X_1X_2
\]

- Nonadditive terms create the interaction

- No interaction
  \[ S_{ij} = 0 \]

- Interaction between $X_i$ and $X_j$
  \[ S_{ij} > 0 \]
Higher-order Sensitivity Indices

When random variables are independent below holds

\[ 1 = \sum_i S_i + \sum_{i<j} S_{ij} + \ldots + S_{1,2,\ldots,d} \]

Consider an example \( Y = g(X_1, X_2, X_3) \)
Total-effect Index

\[ S_i^T = 1 - \frac{\text{Var}_{X_i} \left[ \mathbb{E}_{x_i} [Y | X_i] \right]}{\text{Var}[Y]} \]

Conditioning on all variables but \( X_i \)

\( S_i^T \) accounts for all the interaction effects associated with a variable \( X_i \).
Total-effect Index

- For example, consider a function

\[ Y = g(X_1, X_2, X_3) \]

Total-effect index for \( X_1 \) is

\[ S_1^T = 1 - S_{23} - S_2 - S_3 \]

When the variables are independent

\[ S_1^T = S_1 + S_{12} + S_{13} + S_{123} \]
Consider uncorrelated $X$ distributed within a unit hyper-cube

$$Y = g(X)$$

The function can be expanded as

$$Y = g_0 + \sum_i g_i(X_i) + \sum_{i<j} g_{ij}(X_i, X_j) + g_{12,...,d}(X_1, X_2, ..., X_d)$$

This formula is called ANOVA representation if

$$\int_0^1 g_u(X_u) dx_k = 0, \quad k \in u$$

for any $u \subseteq \{1, 2, ..., d\}$. For example,

$$\int_0^1 g_{ij}(X_i, X_j) dX_i = 0 \quad \text{and} \quad \int_0^1 g_{ij}(X_i, X_j) dX_j = 0$$
Review

1) **Main Sobol Index**

\[ S_i = \frac{\text{Var}[E[Y|X_i]]}{\text{Var}[Y]} \]

- Large: sensitive
- Small: not

\[ E[Y|X_i] \]

2) **Higher Order Sobol Index**

\[ S_{ij} = \frac{\text{Var}[E[Y|X_i, X_j]]}{\text{Var}[Y]} \]

- \( S_i - S_j \)

Includes 'effect' of \( X_i, X_j \), interaction b/w \( X_i, X_j \)

3) **Total Sobol Index**

\[ S_i^T = 1 - \frac{\text{Var}[E[Y|X_i]]}{\text{Var}[Y]} \]

Eg when \( d=3 \)

\[ S_2^T = S_2 + S_{12} + S_{23} + S_{123} \]
**ANOVA - derivation**

Assume:

\[ x \sim \text{uniform}(0,1), \text{ ind.} \]

\[ y = g(x_1, x_2, \ldots, x_d) \]

\[ = g_0 + \sum_i g_i(x_i) + \sum_{i<j} g_{ij}(x_i, x_j) + \cdots + g_{12\ldots d}(x_1, \ldots, x_d) \]

given orthogonality. \( y \) is unique. \( \Rightarrow \) "ANOVA."

**Example:**

\[
\begin{align*}
\int_0^1 g_{235}(x_2, x_3, x_5) \, dx_2 &= 0 \\
\int_0^1 g_{235}(x_2, x_3, x_7) \, dx_3 &= 0 \\
\vdots & \\
\int_0^1 g_{235}(x_2, x_3, x_5) \, dx_5 &= 0
\end{align*}
\]

**Sensitivity**

\[ s_i = \frac{V_i}{V} \]

\[ \text{Variance fraction coming from } x_i \]

\[ \text{interaction. } \]

\[ 50\% \quad x_1 \]

\[ 40\% \quad x_2 \]

\[ 10\% \quad x_1 \text{ & } x_2 \]

\[ \text{Var}[y] = V = \sum_i V_i + \sum_{i<j} V_{ij} + \cdots + V_{12\ldots d} \]
Because of orthogonality

\[ E[Y] = 0 \]
\[ E[Y | X_i] = g_0 + g_i(X_i) \]
\[ E[Y | X_i, X_j] = g_0 + g_i(X_i) + g_j(X_j) + g_{ij}(X_i, X_j) \]

\[ g_i(X_i) = E[Y | X_i] - g_0 \]
\[ g_{ij}(X_i, X_j) = E[Y | X_i, X_j] - E[Y | X_i] - E[Y | X_j] + g_0 \]

\[ \frac{\text{Var}[g_i(X_i)]}{\text{Var}[Y]} = \frac{V_i}{V} = \frac{\text{Var}[E[Y | X_i]]}{\text{Var}[Y]} = s_i^2 \]

ANOVA.

decomposition of total var.

Law of total var.
Analysis of Variance (ANOVA) Decomposition

\[ Y = g_0 + \sum_i g_i(X_i) + \sum_{i<j} g_{ij}(X_i, X_j) + g_{12,..,d}(X_1, X_2, \ldots, X_d) \]

Taking \( \text{Var} [\cdot] \) on both sides

\[ \text{Var}[Y] = V = \sum_i V_i + \sum_{i<j} V_{ij} + \ldots + V_{12..d} \]

The proportion of variance attributed to \( X_i \)

\[ 1 = \sum_i \frac{V_i}{V} + \sum_{i<j} \frac{V_{ij}}{V} + \ldots + \frac{V_{12..d}}{V} \]

Equivalent to Sobol index  why?
Analysis of Variance (ANOVA) Decomposition

\[ Y = g_0 + \sum_i g_i(X_i) + \sum_{i<j} g_{ij}(X_i, X_j) + g_{12,\ldots,d}(X_1, X_2, \ldots, X_d) \]

\[ = 0 \]

\[ \mathbb{E}[\cdot] \text{ on both sides, i.e. integrate over } X \]

\[ \mathbb{E}[Y] = g_0 \]

\[ \mathbb{E}[\cdot | X_u] \text{ on both sides, i.e. integrate over all but } u \subseteq \{1, 2, \ldots, d\} \]

\[ \mathbb{E}_{x_i}[Y|X_i] = g_0 + g_i(X_i) \]

\[ \mathbb{E}_{x_i}[Y|X_i, X_j] = g_0 + g_i(X_i) + g_j(X_j) + g_{ij}(X_i, X_j) \]

\[ \ldots \]
Analysis of Variance (ANOVA) Decomposition

\[ \mathbb{E}_{x_i}[Y|X_i] = g_0 + g_i(X_i) \]
\[ \mathbb{E}_{x_i}[Y|X_i, X_j] = g_0 + g_i(X_i) + g_j(X_j) + g_{ij}(X_i, X_j) \]

\[ \text{Var}_{x_i}[\mathbb{E}_{x_i}[Y|X_i]] = V_i \]
\[ \text{Var}_{x_i}\left[\mathbb{E}_{x_i}[Y|X_i, X_j]\right] = V_i + V_j + V_{ij} \]

\[ \frac{\text{Var}_{x_i}[\mathbb{E}_{x_i}[Y|X_i]]}{\text{Var}[Y]} = \frac{V_i}{V} = S_i \]
\[ \frac{\text{Var}_{x_i}[\mathbb{E}_{x_i}[Y|X_i, X_j]]}{\text{Var}[Y]} = \frac{V_i}{V} + \frac{V_j}{V} + \frac{V_{ij}}{V} = S_i + S_j + S_{ij} \]
ANOVA vs. The Law of Total Variance

Consider $X_1$,

The Law of Total Variance

$$\text{Var}[Y] = \text{Var}_{x_i} \left[ \mathbb{E}_{x_i}[Y|X_1] \right] + \mathbb{E}_{x_i} \left[ \text{Var}_{x_i}[Y|X_1] \right]$$

ANOVA

$$\text{Var}[Y] = V_1 + \sum_{i=2}^{d} V_i + \sum_{i<j} V_{ij} + ... + V_{12...d} = \text{Var}_{x_i} \left[ \mathbb{E}_{x_i}[Y|X_1] \right]$$
Remarks

• When variables are correlated
  • The Law of Total Variance does not require the assumption of independence
    • Intuitive interpretation still holds

\[ 1 > \sum_{i} S_i + \sum_{i<j} S_{ij} + \ldots + S_{1,2,\ldots,d} \]
Remarks

• When \( X \) are independent random variables, the sensitivity indices are invariant to any one-on-one transformation of input \( Z_i = T_i(X_i) \).

One-on-one transform

\[ Z_i = T_i(X_i) \]

\[ Z_1 \quad Z_2 \quad Z_3 \quad Z_4 \]

\[ X_1 \quad X_2 \quad X_3 \quad X_4 \]

\[ Y = g(X) \]

\[ Y \quad \tilde{Y} = aY + b \]

Linear transform

• The sensitivity indices are invariant to the linear transform of output.
Example: FORM limit state function

\[
G_{\text{FORM}}(\mathbf{z}) = \nabla G(\mathbf{z}^*) \cdot (\mathbf{z} - \mathbf{z}^*)
\]

\[
= \frac{\partial G}{\partial z_1}(\mathbf{z}^*) \cdot (z_1 - z_1^*) + \frac{\partial G}{\partial u_2}(\mathbf{z}^*) \cdot (z_2 - z_2^*)
\]

\[
+ \ldots + \frac{\partial G}{\partial z_d}(\mathbf{z}^*) \cdot (z_d - z_d^*)
\]

\[
\text{Var}[Y] = (\frac{\partial G}{\partial z_1}(\mathbf{z}^*))^2 + (\frac{\partial G}{\partial z_2}(\mathbf{z}^*))^2 + \ldots + \frac{\partial G}{\partial z_d}(\mathbf{z}^*)^2
\]

\[
= \|\nabla G\|^2
\]

\[
E[Y|Z_1] = \frac{\partial G}{\partial z_1}(\mathbf{z}^*) \cdot Z_1 + \text{const.}
\]

\[
\text{Var}_{Z_1}[E[Y|Z_1]] = \left(\frac{\partial G}{\partial z_1}(\mathbf{z}^*)\right)^2.
\]

\[
S_i = \left|\frac{\frac{\partial G}{\partial z_1}(\mathbf{z}^*)}{\|\nabla G\|^2}\right|^2 = \alpha_i^2 \leftrightarrow \text{Importance vector.}
\]

\[
\mathbf{z}_i \sim N(0,1) \quad \text{ind.}
\]
Special case – Linear model $g(x)$

- For a linear model, below are equivalent
  - Sigma-normalized derivative
  - Linear regression coefficients
  - Variance-based sensitivity indices

- Example – FORM limit state surface

$$G_{FORM}(z) = \nabla G(z^*)(z - z^*)$$

$$= \frac{\partial G(z^*)}{\partial z_1} (z_1 - z_1^*) + \cdots + \frac{\partial G(z^*)}{\partial z_d} (z_d - z_d^*)$$

$$\text{Var}[Y_{FORM}] = ||\nabla G(z^*)||^2$$

$$E[Y_{FORM}|z_i] = \frac{\partial G(z^*)}{\partial z_i} z_i$$

$$\text{Var}[E[Y_{FORM}|z_i]] = \left(\frac{\partial G(z^*)}{\partial z_i}\right)^2$$

$$S_i = \frac{\text{Var}[E[Y_{FORM}|z_i]]}{\text{Var}[Y_{FORM}]} = \alpha^2$$
Algorithm GSA.

1. MCS

\[ E_i = \frac{1}{N} \sum_{n=1}^{N} y^{(n)} \]

\[ \text{Var}[E_i] = \text{sample var. of } E_i \]

\[ = \frac{1}{M} \sum_{m=1}^{M} (E_i - E_i)^2 \]

\[ \exists N \times M \times d \times 2 \]

2. Smart MCS

\[ E_i^{(m)} \xrightarrow{\text{resample}} y^{A} \text{, } x^{A'} \xrightarrow{\text{resample}} y^{A} \]

3. Probability model-based

\[ f(y|x_i) = \frac{f(x_i, y)}{f(x_i)} \Rightarrow f(x_i, y) \]

\[ \text{fitting joint PDF of } x_i, y \]
Algorithms: (1) Monte Carlo Estimation

Requires two-fold integration for “variance” and “mean” operation

\[ S_i = \frac{\text{Var}_{x_i} \left[ \mathbb{E}_{x_i} [Y | x_i] \right]}{\text{Var}[Y]} \]

For \( n = 1:N \)

sample \( x_i^{(n)} \)

For \( m = 1:N \)

sample \( x_i^{(m)} \)

simulate \( y^{(m,n)} = g \left( x_i^{(n)}, x_i^{(m)} \right) \)

end

\[ E^{(n)} = \mathbb{E}_{x_i} \left[ Y | x_i^{(n)} \right] \approx \frac{1}{N} \sum_{m=1}^{N} y^{(m,n)} \]

end

\[ \text{Var}_{x_i} \left[ \mathbb{E}_{x_i} [Y | x_i] \right] \approx \text{sample variance of } E^{(n)} \]
Algorithms: (2) Smart Monte Carlo

• Start with two random $N$ sample set

\[ A = \{ x^{(1)}, x^{(2)}, \ldots, x^{(N)} \} \]

\[ B = \{ y^{(1)}, y^{(2)}, \ldots, y^{(N)} \} \]

• Designed sample set to estimate Sobol indices of $X_1$

\[ A_B = \{ \tilde{x}_1^{(1)}, \tilde{x}_2^{(1)}, \ldots, \tilde{x}_D^{(1)} \} \]

\[ \tilde{y}^{(1)} \]

Case 1

Designed resampling to evaluate $S_i$

Case 2

Designed resampling to evaluate $S_j$
## Algorithms: (2) Smart Monte Carlo

Saltelli, 2009

| $V_{X_i}(E_{X_{-i}}(Y|X_i))$ for $S_i$ | Reference |
|----------------------------------------|-----------|
| (a) $\frac{1}{N} \sum_{j=1}^{N} f(A)(B_{A}^{(i)})_j - f_0^2$ | ‘Sobol’ 1993’ [37] |
| (b) $\frac{1}{N} \sum_{j=1}^{N} f(B)(A_B^{(i)})_j - f(A)_j$ | [this paper] |
| (c) $V(Y) - \frac{1}{2N} \sum_{j=1}^{N} (f(B)_j - f(A_B^{(i)})_j)^2$ | ‘Jansen 1999’ [14] |

| $E_{X_{-i}}(V_{X_i}(Y | X_{-i}))$ for $S_{Ti}$ | |
|----------------------------------------|-----------|
| (d) $V(Y) - \frac{1}{N} \sum_{j=1}^{N} f(A)_j f(A_B^{(i)})_j + f_0^2$ | ‘Homma 1996’ [11] |
| (e) $\frac{1}{N} \sum_{j=1}^{N} f(A)_j (f(A)_j - f(A_B^{(i)})_j)$ | ‘Sobol’ 2007’ [39] |
| (f) $\frac{1}{2N} \sum_{j=1}^{N} (f(A)_j - f(A_B^{(i)})_j)^2$ | ‘Jansen 1999’ [14] and [this paper] |
Algorithms: (3) Probability model-based GSA

- $N$-MCS samples are required - existing samples can be used!

**Estimation algorithm**

- Approximate joint distribution of $f(X_i, Y)$ using a Gaussian mixture model (GMM)
- Estimate $E[Y|X_i]$ from GMM $f(X_i, Y)$
- Repeat for different $X_i^{(n)}$ samples to get sample variance

\[
\text{Var}_{x_i} \left[ E_{x_i} [Y|X_i^{(n)}] \right]
\]
For Thursday class (4/21)

- Bring laptop
- Download quoFEM
- Register for a DesignSafe Account

- quoFEM: https://simcenter.designsafe-ci.org/research-tools/quofem-application/
- DesignSafe: https://www.designsafe-ci.org/account/register/
Variance-based Reliability Sensitivity Analysis

- Reliability-oriented sensitivity analysis
- Quantity of interest:

\[ q = \mathbb{1}(G(X)) = \begin{cases} 1 & G(X) \leq 0 \\ 0 & G(X) > 0 \end{cases} \]

\[ E[q] = E[\mathbb{1}(G(X))] = P_f \]

\[ \text{Var}[q] = \text{Var}[\mathbb{1}(G(X))] = P_f (1 - P_f) \]

Bernoulli

\[ P(q=1) = P_f \]

\[ P(q=0) = 1 - P_f \]
Reformulation of Sobol index

• Main Sobol index

\[
S_i = \frac{\text{Var}_{X_i} \left[ \mathbb{E}_{X_i}[q|X_i] \right]}{\text{Var}[q]} = \frac{\text{Var}_{X_i} \left[ \mathbb{E}_{X_i}[q|X_i] \right]}{P_f(1 - P_f)}
\]

• Similarly,

\[
\mathbb{E}_{X_i}[q|X_i] = P_f|X_i
\]

\[
\text{Var}_{X_i} \left[ \mathbb{E}_{X_i}[q|X_i] \right] = \text{Var}_{X_i}[P_f|X_i]
\]

\[
= \mathbb{E}_{X_i}[P_f^2|X_i] - \mathbb{E}_{X_i} \left[ \mathbb{E}_{X_i}[P_f|X_i] \right]^2
\]

\[
= \mathbb{E}_{X_i}[P_f^2|X_i] - P_f^2
\]

\[
S_i = \frac{\text{Var}_{X_i} \left[ \mathbb{E}_{X_i}[q|X_i] \right]}{P_f(1 - P_f)} = \frac{\mathbb{E}_{X_i}[P_f^2|X_i] - P_f^2}{P_f(1 - P_f)}
\]
Reformulation of Sobol index

\[ S_i = \frac{\mathbb{E}_{X_i} [P_f^2 | X_i] - P_f^2}{P_f (1 - P_f)} \]

- \( P_f \) is the solution of reliability analysis
- How about \( P_f | X_i \)?

Two different combination of **Reliability Analysis and Variance Based Sensitivity analysis**:

1. Sobol indices as “by-product” of reliability analysis
   - After FORM reliability analysis
   - After sampling-based reliability analysis

2. Get Sobol indices “before” running reliability analysis
   - Probability model-based GSA
Review of FORM - \( \beta \) and \( \alpha \)

\( \beta \): reliability index

\[ P_f = \Phi(-\beta) \]

\[ \beta = ||z^*|| \]

\( \alpha_i \): importance vector

\[ \alpha = \frac{\nabla G(z^*)}{||\nabla G(z^*)||}, \quad ||\alpha|| = 1 \]

\[ \alpha_1 + \alpha_2 + \cdots + \alpha_d = 1 \]

\[ \beta = \alpha \cdot z^* = ||\alpha|| ||z^*|| = 1 \cdot \beta \]

Given \( \beta, \alpha \)

\( f(z) \sim \text{standard normal} \)

\[ G(z) = \nabla G(z^*)(z - z^*) = 0 \]

\[ \alpha (z - z^*) = 0 \]

\[ \alpha z - \alpha z^* = \alpha z - \beta = 0 \]
FORM and Variance-based Sensitivity Analysis

- FORM limit state

\[ G_{FORM}(z) = \nabla G(z^*)(z - z^*) \]

or

\[ G_{FORM}(z) = \beta - \alpha z \]

**Goal:** to derive \( S_i \) in terms of \( \alpha_i \) and \( \beta \)

\[ P_f = \mathbb{P}(\alpha Z \geq \beta) = \mathbb{P}(\alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_d Z_d \geq \beta) = \mathbb{P}(\tilde{Z} \geq \beta) = \Phi(-\beta) \]

\[ S_i = \frac{\mathbb{E}_{X_i}[P_f^2 \mid X_i] - P_f^2}{P_f(1 - P_f)} \]

Standard normal
FORM and Variance-based Sensitivity Analysis

Goal: to derive $S_i$ in terms of $\alpha$ and $\beta$

\[ P_f = \mathbb{P}(\alpha Z > \beta) = \mathbb{P}(\alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_d Z_d > \beta) = \mathbb{P}(\tilde{Z} > \beta) = \Phi(-\beta) \]

\[ P_{f|Z_i} = \mathbb{P}(\alpha_i Z_i > \beta - \alpha_i z_i) = \mathbb{P}\left(\tilde{Z} > \frac{\beta - \alpha_i z_i}{\|\alpha_i\|}\right) = \Phi\left(-\frac{\beta - \alpha_i z_i}{\|\alpha_i\|}\right) \]

\[ S_i = \frac{\mathbb{E}_{X_i}[P_f^2|X_i] - P_f^2}{P_f(1 - P_f)} \]
FORM and Variance-based Sensitivity Analysis

\[
\mathbb{E}_{Z_i}[P^2_f|Z_i] = \mathbb{E}_{Z_i}[P_f|Z_i P_f|Z_i] \\
= \mathbb{E}_{Z_i} \left[ \Phi \left( \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right) \Phi \left( \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right) \right] \\
= \mathbb{E}_{Z_i} \left[ P_{\tilde{Z}} \left[ \tilde{Z}_1 \leq \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right] P_{\tilde{Z}} \left[ \tilde{Z}_2 \leq \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right] \right] \\
= \mathbb{E}_{Z_i} \left[ P_{\tilde{Z}_i,\tilde{Z}} \left[ \left( \tilde{Z}_1 \leq \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right) \cap \left( \tilde{Z}_2 \leq \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right) \right] \right] \\
= P_{\tilde{Z}_i,\tilde{Z}} \left[ \left( \tilde{Z}_1 \leq \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right) \cap \left( \tilde{Z}_2 \leq \frac{\alpha_i Z_i - \beta}{\|\alpha_i\|} \right) \right] \\
= P \left[ (\tilde{Y}_1 \leq -\beta) \cap (\tilde{Y}_2 \leq -\beta) \right] = \Phi_2(-\beta, -\beta; \alpha^2_i) \\
\tilde{Y}_1 = \tilde{Z}_1 \|\alpha_i\| - \alpha_i Z_i \quad \tilde{Y}_1 \sim N(0,1)^2 \\\n\tilde{Y}_2 = \tilde{Z}_2 \|\alpha_i\| - \alpha_i Z_i \quad \tilde{Y}_2 \sim N(0,1)^2 \quad \text{corr}[\tilde{Y}_1, \tilde{Y}_2] = \alpha^2_i \]

\[
S_i = \frac{\mathbb{E}_{X_i}[P^2_f|X_i] - P^2_f}{P_f(1 - P_f)} 
\]
FORM and Variance-based Sensitivity Analysis

- **Main-effect Sobol index**

\[
S_i = \frac{\mathbb{E}_{Z_i}[P_{f|Z_i}^2] - P_f^2}{P_f(1 - P_f)} = \frac{\Phi_2(-\beta, -\beta; \alpha_i^2) - P_f^2}{P_f(1 - P_f)} = \frac{1}{P_f(1 - P_f)} \int_0^{\alpha_i^2} \varphi_2(-\beta, -\beta; r)dr
\]

- **Total-effect Sobol index**

\[
S_i^T = 1 - \frac{\mathbb{E}_{Z_i}[P_{f|Z_i}^2] - P_f^2}{P_f(1 - P_f)} = 1 - \frac{\Phi_2(-\beta, -\beta; \|\alpha_i\|^2) - P_f^2}{P_f(1 - P_f)}
\]

\[
= \frac{1}{P_f(1 - P_f)} \int_{1-\alpha_i^2}^{1} \varphi_2(-\beta, -\beta; r)dr
\]

\[
\Phi_2(-\beta, -\beta, \|\alpha_v\|^2) = \Phi(-\beta)^2 + \int_0^{\|\alpha_v\|^2} \varphi_2(-\beta, -\beta, r)dr.
\]
Example with two Random Variables

- **Effect of $\alpha$**
  - $s_1 = 1$
  - $s_2 = 0$
  - $s_{12} = 0$

- **Effect of $\beta$**
  - Interaction
  - Smaller $\beta$
  - Larger $\beta$

- $\alpha = [1, 0]$

- $s_1 = s_2$
- $s_1^T = s_2^T$
- $s_{12} = ??$ (depends on $\beta$)

- Larger interaction as $\beta \to \infty$, $\begin{cases} s_1 \to 0 \\ s_1^T \to 1 \end{cases}$
FORM and Variance-based Sensitivity Analysis

(Papaioannou and Straub, 2021)
Sampling-based Reliability Analysis and $S_i$

1. collect $X_F^{(n)}$ (and $W^{(n)}$)
2. fit distribution of $f(X|F)$ using (1)
3. calculate $S_i, S_i^T$ using $f(X|F)$
Sampling-based Reliability Analysis and $S_i$

- Again, reformulation of Sobol index

$$P_{f|X_i} = \mathbb{P}(F|X_i)$$

$$= \frac{\mathbb{P}(X_i|F) \mathbb{P}(F)}{\mathbb{P}(X_i)}$$

$$= \frac{\int_{-\infty}^{\infty} f_{X_i|F}(X_i) dX_i P_f}{f(X_i) dX_i}$$

$$= \frac{f_{X_i|F}(X_i) P_f}{f(X_i)}$$

$$S_i = \frac{\mathbb{E}_{X_i}[P^2_{f|X_i}] - P_f^2}{P_f(1-P_f)}$$
Sampling-based Reliability Analysis and $S_i$

- Approximation of $f_{X_i|\mathcal{F}}(X_i)$ using kernel density estimation or cross entropy-based distribution fitting

Kernel density estimation

Mixture distribution fitting

- Estimation of total-effect index is more challenging
Sensitivity Analysis before Reliability Analysis

- Probability model-based approach
- Let us define
  \[ Y = G(X) \]
  
  \[ P_{f|X_i} = P(Y \leq 0|X_i) = \frac{f(X_i, Y \leq 0)}{f(X_i)} \]

  \[ f(X_i, Y \leq 0) = \int_{-\infty}^{0} f_{X_i,Y}(X_i, Y) dY \]

  \[ f(X_i) = \int_{-\infty}^{\infty} f_{X_i,Y}(X_i, Y) dY \]

- Approximate joint distribution of \( f(X_i, Y) \) using a Gaussian mixture model (GMM)
- The mixture model “extrapolates” the samples
  → Not accurate for rare events

\[ S_i = \frac{\mathbb{E}_{X_i} \left[ P_{f|X_i}^2 - P_f^2 \right]}{P_f(1 - P_f)} \]
Toy Example

\[ Y = \frac{2p}{Ebh^3} \left[ (4 + 5\nu) \frac{h^2L}{4} + 2L^3 \right], \]

Fig. 11. A cantilever beam.

Table 1
Random variables of the cantilever beam example.

<table>
<thead>
<tr>
<th>Variable</th>
<th>p (kN)</th>
<th>(\nu)</th>
<th>b (m)</th>
<th>h (m)</th>
<th>L (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution</td>
<td>Normal</td>
<td>Normal</td>
<td>Lognormal</td>
<td>Lognormal</td>
<td>Lognormal</td>
</tr>
<tr>
<td>Mean</td>
<td>65</td>
<td>0.225</td>
<td>0.2</td>
<td>0.3</td>
<td>1.5</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.5</td>
<td>0.03</td>
<td>0.02</td>
<td>0.02</td>
<td>0.05</td>
</tr>
</tbody>
</table>

\[ f_E(e) = 0.4\mathcal{N}(e, 200, 1) + 0.6\mathcal{N}(e, 190, 0.95^2), \]
Truss Model

Input variables

\( x_1, x_2: \) load1 \((P_1)\) and load2 \((P_1)\) with (correlation 0.6)

\( x_3 \sim x_{27}: \) strength of each member, lognormal

Output variable: limit state function

\[ g(x) = \min_{k=1,...,25} \left( \sigma_k^{\text{thr}} - |\sigma_k(P_1, P_2)| \right) \]
Examples

• Structural model: Shear building (Opensees)

• Input parameters

<table>
<thead>
<tr>
<th>Name</th>
<th>Mean</th>
<th>C.O.V</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>100</td>
<td>0.1</td>
</tr>
<tr>
<td>wR</td>
<td>50</td>
<td>0.1</td>
</tr>
<tr>
<td>k</td>
<td>326</td>
<td>0.1</td>
</tr>
<tr>
<td>Fy</td>
<td>50</td>
<td>0.1</td>
</tr>
<tr>
<td>alpha</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>factor (PGA)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

• Excitation

Rinaldi near-field
Nonlinear behavior

- Hysteresis curves for Rinaldi UQ
Examples

node2 disp

node2 acc

node7 disp

node7 acc
“(...) Parameters 3, 6, 7, 8, 13, and 14 should be excluded from the parameter candidates because they have little influence over the objective function.”

Examples:
Reducing the complexity of multi-objective optimization

Multi-objective Water Distribution System Optimization

- 21 pipes in a system
- 16 Retrofitting options of each pipe:
  - 15 available diameters ranging 0.914 - 5.182 m
  - Do nothing
- Conflicting objectives:
  - Cost: capital cost (pipes, tanks, and pumps) + operating cost during a design period
  - Performance:
    - eg. surplus power energy per unit weight

New York Tunnels Rehabilitation
(21 components)

Goal: **Maximize** the performance, **minimize** the cost

Examples: Multi-objective Water Distribution System Optimization